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DISTANCE SPECTRUM OF TWO FAMILIES OF GRAPHS

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Abstract. Let H_1 and H_2 be two copies of the complete graph K_n, $n \geq 3$ with vertex sets $V(H_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_n\}$. Graph $\Gamma(n,p), 1 \leq p \leq n-1$, is obtained from the union of graphs H_1 and H_2 by adding edges $\{u_i v_i | i \in \{1, 2, \ldots, p\}\}$. Graph $\Theta(n)$ is obtained from the union of graphs H_1 and H_2 by joining each vertex v_i of H_1 to every vertex in $\{u_1, u_2, \ldots, u_n\} \setminus \{u_i\}, i = 1, 2, \ldots, n$. The adjacency spectrum of $\Gamma(n,p)$ and $\Theta(n)$ were determined in [9]. An open problem posed in [7] was to find families of graphs of diameter greater than two, for which the adjacency and distance spectrum are both integral. To answer the open problem, the distance spectrum of the above family of graphs is calculated, and new distance equienergetic graphs are constructed in this paper.

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1. Introduction

Spectral graph theory studies various aspects of the spectra of the matrices associated with graphs. Originally, spectral graph theory analyzed the adjacency matrix of a graph, especially its eigenvalues. Collatz and Sinogowitz first began the exploration of this topic in 1957 [1]. A significant result involving the distance spectrum was a contribution by Graham and Pollak [4] while studying data communication problems. Two detailed surveys of distance spectrum are seen in [12] and in [13].

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Distance matrix of G is defined as $\mathcal{D}(G) = (d_{ij})$, a symmetric matrix, where d_{ij} is the distance between the vertices v_i and v_j. The characteristic polynomial of $\mathcal{D}(G)$ is given by $\det(xI - \mathcal{D}(G))$, where $xI - \mathcal{D}(G)$ is an invertible matrix. The real eigenvalues of $\mathcal{D}(G)$ are the $\mathcal{D}$-eigenvalues of G and form the $\mathcal{D}$-spectrum of G, denoted by $spec_{\mathcal{D}}(G)$. If there are t distinct $\mathcal{D}$-eigenvalues, say $\mu_1, \mu_2, \ldots, \mu_t$, with multiplicities α_1, α_2, ..., α_t, respectively, then $spec_{\mathcal{D}}(G)$ is written as $\{\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots \mu_t^{\alpha_t}\}$ or $\begin{pmatrix} \mu_1, & \mu_2, & \ldots & \mu_t \\ \alpha_1 & \alpha_2 & \ldots & \alpha_t \end{pmatrix}$ as per convenience. Two graphs are said to be cospectral if they have the same spectra. The $\mathcal{D}$-energy [8] is the sum of the absolute values of the $\mathcal{D}$-eigenvalues of G. Two graphs G and H of the same order are considered distance equienergetic if the distance energy of the graphs G and H are equal. Several non-cospectral distance equienergetic graphs have been constructed in [7], [8], [10] and [11].

Join $G \triangledown H$ of two vertex disjoint graphs G and H is a simple graph obtained from their union $G \bigcup H$ by adding edges between every vertex of G and every vertex of H. An integral graph is a graph that has only integer eigenvalues. It is proved in [8] that for an r- r-regular graph of diameter 2, the adjacency spectrum is integral if and only if the distance spectrum is integral. An open problem posed in [7] was finding families of graphs of diameter greater than two, for which the adjacency and distance spectrum are integral.

Two families of graphs, namely $\Gamma(n,p)$ and $\Theta(n)$, were defined, and their adjacency spectrum was determined in [9]. As an attempt to answer the open problem, the distance spectrum of $\Gamma(n,p)$ and

$\Theta(n)$ are found in this paper. Distance non-cospectral graphs that are also distance equienergetic are discussed.

Throughout the paper, the following notations will be used.

I_n: identity matrix of order n.
J_n: all one square matrix of order n.
$J_{s\times t}$: all ones matrix of order $s \times t$.
$spec_{\mathcal{A}}(G)$: adjacency spectrum of graph G.
diam(G): diameter of G.
det(S): determinant of matrix S.
$\overline{G}$: complement of G.
L(G): line graph of G.
$spec_{\mathcal{D}}(G)$: distance spectrum of graph G.
$E_{\mathcal{D}}$: $\mathcal{D}$- energy.
$G \bigcup H$: union of the graphs G and H.
R^T: transpose of incidence matrix R.

2. Preliminaries

Lemma 2.1. [2] Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are square matrices, and A is an invertible matrix. Then $\det(S) = \det(A)\det(D - CA^{-1}B)$.

Lemma 2.2. [3] Let $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a symmetric 2×2 block matrix. Then, the spectrum of A is the union of the spectrum of $A_0 + A_1$ and $A_0 - A_1$.

Lemma 2.3. [2] Let G be a regular graph of order p with an adjacency matrix A and an incidence matrix R. Let $L(G)$ be its line graph. Then $R^TR = A(L(G)) + 2I$.

Lemma 2.4. [2] If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G with adjacency matrix A, then $\det A = \prod_{i=1}^{p}\lambda_i$. Also for any polynomial $P(x), P(\lambda)$ is a characteristic value of $P(A)$ and hence $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2.5. [2] Let G be a connected r-regular graph on p vertices with its adjacency matrix A having m distinct eigenvalues $\lambda_1 = r, \lambda_2, \ldots, \lambda_m$. Then there exists a polynomial $P(x) = p\dfrac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\cdots(r-\lambda_m)}$ such that $P(A) = J$ where J is the square matrix of order p whose all entries are one, so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.

Lemma 2.6. [2] Let G be an r-regular graph on p vertices. Then $\overline{G}$ is $p - r - 1$ regular and $L(G)$ is $2r - 2$ regular. If $\{r, \lambda_2, \ldots, \lambda_p\}$ are the adjacency eigenvalues of G, then

(1) The adjacency eigenvalues of $\overline{G}$ are $p - r - 1$ and $-\lambda_i - 1, i = 2, 3, \ldots, p$.
(2) The adjacency eigenvalues of $L(G)$ are $2r-2, \lambda_i + r - 2, i = 2, 3, \ldots, p$ and -2 with multiplicity $\dfrac{p(r-2)}{2}$.

Lemma 2.7. [8] Let G be an r-regular graph of order p and $diam(G) = 2$. If $\{r, \lambda_2, \ldots, \lambda_p\}$ are its adjacency eigenvalues, then its $\mathcal{D}$-eigenvalues are $2p - r - 2$ and $-(\lambda_i + 2), i = 2, 3, \ldots, p$.

Lemma 2.8. [5] Let G be an r-regular graph on p vertices. Let $\{r, \lambda_2, \ldots, \lambda_p\}$ be the adjacency eigenvalues of G. If $diam(\overline{G}) = 2$, then the $\mathcal{D}$-eigenvalues of $\overline{G}$ are $\{p + r - 1, \lambda_2 - 1, \ldots, \lambda_p - 1\}$.

Theorem 2.1. [14] *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and spectrum of the adjacency matrix A_{G_i} be $\{r_i, \lambda_{ij} \mid j = 2, \ldots, n_i\}$. The distance spectrum of $G_1 \nabla G_2$ consists of $-\lambda_{ij} - 2$ for $i = 1, 2$, $j = 2, 3, \ldots, n_i$ and two more eigenvalues of the form $n_1 + n_2 - 2 - \frac{r_1+r_2}{2} \pm \sqrt{(n_1 - n_2 - \frac{r_1-r_2}{2})^2 + n_1 n_2}$.*

Theorem 2.2. [14] *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices, whose smallest eigenvalue of the adjacency matrix is at least -2 and that G_i is non-isomorphic to K_n. Then $E_{\mathcal{D}}(G_1 \nabla G_2) = 4(n_1 + n_2) - 2(r_1 + r_2) - 8$.*

3. Distance spectrum of $\Gamma(n, p)$

Definition 3.1. [9] Let H_1 and H_2 be two copies of the complete graph K_n, $n \geq 3$ with vertex sets $V(H_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_n\}$. Graph $\Gamma(n, p), 1 \leq p \leq n - 1$, is a graph obtained from the union of graphs H_1 and H_2 by adding edges $\{u_i v_i | i \in \{1, 2, \ldots, p\}\}$.

When $p = n$, $spec_{\mathcal{A}}(\Gamma(n, n) = \{n, n-2, 0^{n-1}, -2^{n-1}\}$.

Theorem 3.1. *For $n \geq 3$ and $1 \leq p \leq n - 1$, the spectrum of the distance matrix of $\Gamma(n, p)$ is given as*

$$spec_{\mathcal{D}}(\Gamma(n,p)) = \begin{cases} -1, & 2(n-p-1) \text{ times} \\ -2, & p-1 \text{ times} \\ 0, & p-1 \text{ times} \\ \text{2 roots of} & x^2 + (2n+1-p)x + p(n-p+1) = 0 \\ \text{2 roots of} & x^2 - (4n-p-3)x + (n-p)(3p-8) - (3p-2) = 0. \end{cases}$$

Proof. With the proper labelling of vertices, the distance matrix of $\Gamma(n, p)$ is given as

$$\mathcal{D}(\Gamma(n,p)) = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}_{2n \times 2n} \text{ where}$$

$$A_0 = \begin{pmatrix} (J-I)_p & J_{p \times n-p} \\ J_{n-p \times p} & (J-I)_{n-p} \end{pmatrix}_{n \times n} \text{ and } A_1 = \begin{pmatrix} (2J-I)_p & 2J_{p \times n-p} \\ 2J_{n-p \times p} & 3J_{n-p} \end{pmatrix}_{n \times n}$$

The $\mathcal{D}$- spectrum can be calculated using Lemma 2.2 as follows.

$$\text{Let} \quad B = A_0 - A_1 = \begin{pmatrix} -J & -J \\ -J & -(2J+I) \end{pmatrix}_{n \times n}.$$

the characteristic polynomial of B,

$$
\begin{aligned}
\det(xI_n - B) &= \det\begin{pmatrix} xI+J & J \\ J & (x+1)I+2J \end{pmatrix} \\
&= \det\Big(xI+J\Big)\ \det\Big((x+1)I+2J-J(xI+J)^{-1}J\Big)\text{, by Lemma 2.1} \\
&= x^{p-1}(x+p)\ \det\Big((x+1)I+(2-\frac{p}{x+p})J\Big) \\
&= \frac{x^{p-1}}{(x+p)^{n-p-1}}\ \det\Big((x+1)(x+p)I+(2x+p)J\Big) \\
&= \frac{x^{p-1}}{(x+p)^{n-p-1}}(x+1)^{n-p-1}(x+p)^{n-p-1}\left[(x+1)(x+p)+(2x+p)(n-p)\right] \\
&= x^{p-1}(x+1)^{n-p-1}\left[(x^2+(2n+1-p)x+p(n-p+1)\right].
\end{aligned}
$$

Similarly, the characteristic polynomial of $C = A_0 + A_1$ is determined as

$$\det(xI_n - C) = (x+1)^{n-p-1}(x+2)^{p-1}\left[(x^2-(4n-p-3)x+(3np-8n-3p^2+5p+2)\right]$$

Hence the result is obtained. □

Corollary 3.1. $E_{\mathcal{D}}(\Gamma(n,p)) =$
$|-1|2(n-p-1)+|-2|(p-1)+2n+1-p+\sqrt{(4n-p-3)^2-4(n-p)(3p-8)-(3p-2)}$

$$= 4n-p-3+\sqrt{(4n-p-3)^2-4((n-p)(3p-8)-(3p-2))}$$

□

Corollary 3.2. $E_{\mathcal{D}}(\Gamma(n,n))$= $6n-4$, for all $n \geq 3$.

Proof. The distance spectrum of $\Gamma(n,n)$ consists of -2 and 0 repeated $n-1$ times each, -n and 3n-2. Hence the result is obtained. □

Theorem 3.2. *$\overline{\Gamma(n,n)}$, $n \geq 3$ is an $\mathcal{A}$- integral as well as a $\mathcal{D}$- integral graph.*

Proof. $\overline{\Gamma(n,n)}$ is a graph of diameter 3. Distance matrix of $\overline{\Gamma(n,n)}$ is given as

$$
\begin{aligned}
\mathcal{D}(\overline{\Gamma(n,n)}) &= \begin{pmatrix} 2J-2I & J+2I \\ J+2I & 2J-2I \end{pmatrix}_{2n\times 2n} \\
&= \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}
\end{aligned}
$$

where $A_0 = 2J-2I,\ A_1 = J+2I$.
Applying Lemma 2.2, $spec_{\mathcal{D}}(\overline{\Gamma(n,n)}) = \left\{3n, n-4, 0^{n-1}, -4^{n-1}\right\}$
and $spec_{\mathcal{A}}(\overline{\Gamma(n,n)}) = \{\pm(n-1), \pm1^{n-1}\}$ obtained by applying Lemma 2.6. □

The above theorem answers the open problem in [7] to find families of graphs of diameter greater than two for which the adjacency and distance spectrum are integral.

Corollary 3.3. $E_{\mathcal{D}}(\overline{\Gamma(n,n)}) =$

$$\begin{cases} 18 & , n=3 \\ 8(n-1) & , \forall n \geq 4 \end{cases}$$

□

Theorem 3.3. *The distance spectrum of line graph of* $\Gamma(n,n)$, $n \geq 3$ *is*

$$spec_{\mathcal{D}}(L(\Gamma(n,n))) = \begin{cases} 1 & , \dfrac{n(n-3)}{2} \textit{ times} \\ -1 & , \binom{n}{2}-1 \textit{ times} \\ -\binom{n}{2}-1 & , \textit{once} \\ \textit{two roots of } x^2+(2n-3)x+2(n-3) & , n-1 \textit{ times} \\ \textit{two roots of } 2x^2-(5n^2-9n+6)x+2(n^3-6n^2+11n-6)=0. \end{cases}$$

Proof. By proper labelling of vertices of $L(\Gamma(n,n))$, the distance matrix of $L(\Gamma(n,n)$ is given as

$$\mathcal{D}(L(\Gamma(n,n))) = \begin{pmatrix} 2J-2I & 2J-R & 2J-R \\ 2J-R^T & 2(J-I)-B & 3J-I-B \\ 2J-R^T & 3J-I-B & 2(J-I)-B \end{pmatrix}, \text{where } B = A(L(K_n))$$

and R^T is the transpose of incidence matrix R.

The characteristic polynomial of $\mathcal{D}(L(\Gamma(n,n)))$

$$= \det \begin{pmatrix} xI_n-2J+2I & R-2J & R-2J \\ R^T-2J & xI_{\binom{n}{2}}-2J+2I+B & I_{\binom{n}{2}}-3J+B \\ R^T-2J & I_{\binom{n}{2}}-3J+B & xI_{\binom{n}{2}}-2J+2I+B \end{pmatrix}$$

Applying elementary transformations and by Lemma 2.1,

(1) $$\det \begin{pmatrix} (x+2)I_n-2J & R-2J & 0 \\ R^T-2J & (x+2)I_{\binom{n}{2}}-2J+B & -((x+1)I_{\binom{n}{2}}+J) \\ 0 & -((x+1)I_{\binom{n}{2}}+J) & 2((x+1)I_{\binom{n}{2}}+J) \end{pmatrix} = \det R \det S$$

where

(2) $$\det R = \det((x+2)I_n-2J) = (x+2)^{n-1}(x-2(n-1))$$

and

$$S = \begin{pmatrix} (x+2)I_{\binom{n}{2}}-2J+B & -((x+1)I_{\binom{n}{2}}+J) \\ -((x+1)I_{\binom{n}{2}}+J) & 2((x+1)I_{\binom{n}{2}}+J) \end{pmatrix} - \begin{pmatrix} R^T-2J \\ 0 \end{pmatrix} \Big((x+2)I_n-2J\Big)^{-1} \Big(R-2J \quad 0\Big).$$

Thus, $\begin{pmatrix} R^T-2J \\ 0 \end{pmatrix} \Big((x+2)I_n-2J\Big)^{-1} \Big(R-2J \quad 0\Big)$

$$= \begin{pmatrix} R^T-2J \\ 0 \end{pmatrix} \frac{1}{(x+2)(x-2(n-1))}(2J+(x-2(n-1))I_n) \Big(R-2J \quad 0\Big)$$

$$= \begin{pmatrix} \dfrac{1}{(x+2)(x-2(n-1))}\big((x-2(n-1))R^TR+4(nx+2n-2x-2)J\big) & 0 \\ 0 & 0 \end{pmatrix}.$$

Substituting in S and applying elementary transformations, we get,

$$S = \begin{pmatrix} 2((x+1)I_{\binom{n}{2}}+J) & -((x+1)I_{\binom{n}{2}}+J) \\ -((x+1)I_{\binom{n}{2}}+J) & xI_{\binom{n}{2}}+\left(\dfrac{x+1}{x+2}\right)R^TR-\dfrac{2x^2}{(x+2)(x-2(n-1))}J \end{pmatrix}$$

Applying Lemma 2.1,

$$\det S = 2^{\binom{n}{2}}(x+1)^{\binom{n}{2}-1}\left(x+\binom{n}{2}+1\right)\det V \tag{3}$$

where

$$V = xI_{\binom{n}{2}} + \left(\frac{x+1}{x+2}\right)R^TR - \frac{2x^2}{(x+2)(x-2(n-1))}J- \\ ((x+1)I_{\binom{n}{2}}+J)\left(2\left((x+1)I_{\binom{n}{2}}+J\right)\right)^{-1}((x+1)I_{\binom{n}{2}}+J)$$

$$= \left(\frac{x-1}{2}\right)I_{\binom{n}{2}} + \left(\frac{x+1}{x+2}\right)R^TR - \left(\frac{5x^2-2(n-2)x-4(n-1)}{2(x+2)(x-2n+2)}\right)J$$

Since $R^TR = A(L(K_n)) + 2I$ and $L(K_n)$ is a $2(n-2)$-regular graph on $\binom{n}{2}$ vertices, by applying Lemma 2.4,

$$\det V = \frac{1}{2^{\binom{n}{2}}(x+2)^{\binom{n}{2}}(x-2(n-1))^{\binom{n}{2}}}\times \\ \prod_{i=1}^{\binom{n}{2}-1}\left[(x^2+5x+2)(x-2(n-1))+2(x+1)(x-2(n-1))\lambda_i\right]\times \\ \left[(x^2+5x+2)(x-2(n-1))+4(x+1)(x-2(n-1))(n-2)-\binom{n}{2}(5x^2-2(n-2))x-4(n-1)\right]$$

where $\lambda_i = \begin{cases} n-4, & n-1 \text{ times} \\ -2, & \dfrac{n(n-3)}{2} \text{ times} \end{cases}$ are the eigenvalues of $L(K_n)$.

$$\therefore \det V = \frac{1}{2^{\binom{n}{2}}(x+2)^{\binom{n}{2}}(x-2(n-1))^{\binom{n}{2}}}(x-2(n-1))^{\binom{n}{2}-1}\left[x^2+(2n-3)x+2(n-3)\right]^{n-1}.$$

$$[(x-1)(x+2)]^{\frac{n(n-3)}{2}}\left[(x+2)(2x^2-(5n^2-9n+6)x+2(n^3-6n^2+11n-6)\right]. \tag{4}$$

Substituting equations (2), (3), (4) in (1), the characteristic polynomial of $\mathcal{D}(L(\Gamma(n,n)))$

$$= (x+1)^{\binom{n}{2}-1}(x+\binom{n}{2}+1)(x-1)^{\frac{n(n-3)}{2}}[x^2+(2n-3)x+2(n-3)]^{n-1}\times \\ [2x^2-(5n^2-9n+6)x+2(n^3-6n^2+11n-6)].$$

Hence the proof is obtained. □

Corollary 3.4. The distance spectrum of the complement of the line graph of $\Gamma(n,n)$ is given as

$$spec_{\mathcal{D}}(\overline{L(\Gamma(n,n))}) = \begin{pmatrix} (n-1)(n+3), & 2n-5, & n-3, & n-5, & -3 \\ 1 & 1 & n-1 & n-1 & n(n-2) \end{pmatrix}.$$

Proof. $L(\Gamma(n,n))$ is a $2(n-1)$-regular graph on n^2 vertices with

$$spec_{\mathcal{A}}(L(\Gamma(n,n))) = \begin{pmatrix} 2(n-1), & n-4, & n-2, & 2(n-2), & -2 \\ 1 & n-1 & n-1 & 1 & n(n-2) \end{pmatrix} \text{ by Lemma 2.6}$$

and since $diam(\overline{L(\Gamma(n,n))}) = 2$, the result holds true by Lemma 2.8. □

4. Distance spectrum of $\Theta(n)$

Definition 4.1. [9] Let H_1 and H_2 be two copies of the complete graph K_n , $n \geq 3$ with vertex sets $V(H_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_n\}$. Graph $\Theta(n)$ is a graph obtained from the union of graphs H_1 and H_2 by joining each vertex v_i to vertices in $\{u_1, u_2, \ldots, u_n\} \backslash \{u_i\}, i = 1, 2, \ldots, n$.
$spec_{\mathcal{A}}(\Theta(n) = \{2(n-1), 0^n, -2^{n-1}\}$.

Theorem 4.1. *The distance spectrum of* $\Theta(n)$ *consists of* $2n$ *once,* 0 *repeated* $n-1$ *times and* -2 *repeated* n *times. Also* $E_{\mathcal{D}}(\Theta(n)) = 4n$.

Proof. The graph $\Theta(n)$ is $2(n-1)$ regular with $2n$ vertices and $diam(\Theta(n)) = 2$.
Since

$$spec_{\mathcal{A}}\Theta(n) = \{2(n-1), 0^n, -2^{n-1}\}.$$

By applying Lemma 2.7, the result holds true. □

Corollary 4.1. $E_{\mathcal{D}}(\overline{\Theta(n)}) = 2n$

Proof. Note that $\overline{\Theta(n)} = nK_2$, the union of n copies of K_2. Hence, $spec_{\mathcal{D}}(\overline{\Theta(n)}) = \{-1^n, 1^n\}$. □

Corollary 4.2. Distance spectrum of line graph of $\Theta(n)$ is given as

$$spec_{\mathcal{D}}(L(\Theta(n))) = \begin{pmatrix} 4(n-1)^2, & 0, & -2(n-2), & -2(n-1) \\ 1 & 2n(n-2) & n-1 & n \end{pmatrix}$$

and $E_{\mathcal{D}}(L(\Theta(n))) = 8(n-1)^2$.

Proof. The results hold by applying Lemmas 2.6 and 2.7. □

Corollary 4.3. The distance spectrum of $\overline{L(\Theta(n))}$ is given as

$$spec_{\mathcal{D}}(\overline{L(\theta(n))}) = \begin{pmatrix} 2n^2+2n-7, & 2n-5, & 2n-7, & -3 \\ 1 & n & n-1 & 2n(n-2) \end{pmatrix}$$

□

5. Distance equienergetic graphs

Theorem 5.1. $\Gamma(n,n)\nabla\Gamma(m,m)$ *and* $\Gamma(n,n)\bigcup\Gamma(m,m)$ *are non-cospectral distance equienergetic graphs.*

Proof. $spec_{\mathcal{D}}(\Gamma(n,n)\bigcup\Gamma(m,m)) = spec_{D}(\Gamma(n,n))\bigcup spec_{\mathcal{D}}(\Gamma(m,m))$

$$= \{-2^{n+m-2}, -n, -m, 0^{n+m-2}, 3n-2, 3m-2\}.$$

From Theorem 2.1,

$$spec_{\mathcal{D}}(\Gamma(n,n)\nabla\Gamma(m,m)) = \{-n, -m, -2^{n+m-2}, 0^{n+m-2}, \frac{3n+3m-4}{2} \pm \frac{1}{2}\sqrt{9(n^2+m^2)-2nm}\}$$

showing that the two graphs are non-cospectral. But,

$$E_{\mathcal{D}}(\Gamma(n,n)\nabla\Gamma(m,m)) = E_{\mathcal{D}}(\Gamma(n,n)\bigcup\Gamma(m,m)) = 6(n+m)-8.$$

□

Theorem 5.2. *$L(\Theta(n))\nabla L(\Theta(m))$ and $L(\Theta(n))\cup L(\Theta(m))$ are non-cospectral distance equienergetic graphs.*

Proof. The proof is obtained in a similar manner to the above theorem and

$$E_{\mathcal{D}}(L(\Theta(n))\nabla L(\Theta(m))) = 8(n-1)^2 + 8(m-1)^2.$$

□

Observation 5.1. For constant $n+m$, $n, m \geq 3$, the following sets of graphs are $\mathcal{D}$-equienergetic graphs:

(1) $\Gamma(n,n)\nabla\Gamma(m,m)$, where $E_{\mathcal{D}}(\Gamma(n,n)\nabla\Gamma(m,m)) = 6(n+m)-8$.
(2) $\Theta(n)\nabla\Theta(m)$, where $E_{\mathcal{D}}(\Theta(n)\nabla\Theta(m)) = 4(n+m)$.

6. Conclusion

The distance spectrum of $\Gamma(n,p)$ and $L(\Gamma(n,n))$ are calculated. $\overline{\Gamma(n,n)}$, a graph of diameter 3, was found to be both adjacency and distance integral. $\overline{\Gamma(n,n)}$ is an answer to the open problem posed in [7]. Families of distance equienergetic graphs are also constructed.

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